6 Patterns, sequences and series

CHAPTER OBJECTIVES:

1.1 Arithmetic sequences and series; sum of finite arithmetic series; geometric sequences and series; sum of finite and infinite geometric series. Sigma notation.

Applications

1.3 The binomial theorem: expansion of $(a+b)^n$, $n \in \mathbb{N}$;

Calculation of binomial coefficients using Pascal's triangle and $\binom{n}{r}$

Before you start

You should know how to:

1 Solve linear and quadratic equations and change the subject of a formula.

e.g. Solve the equation $n(n-4) = 12$

$n^2 - 4n = 12$

$n^2 - 4n - 12 = 0$

$(n-6)(n+2) = 0$

$n = -2, n = 6$

e.g. Make b the subject of this formula.

$ac = b - 3$

$b = ac + 3$

2 Substitute known values into formulae.

e.g. Using the formula $A = 3p^4 - 10q$, find the value of A if $p = 2$ and $q = 1.5$

$A = 3p^4 - 10q$

$A = 3(2)^4 - 10(1.5)$

$A = 3(16) - 15$

$A = 48 - 15$

$A = 33$

Skills check

1 Solve each equation.

a $3x - 5 = 5x + 7$

b $p(2-p) = -15$

c $2^n + 9 = 41$

2 Solve for k.

a $6m + 8k = 30$

b $2pk - 5 = 3$

3 If $T = 2x(x + 3y)$, then find the value of T when

a $x = 3$ and $y = 5$

b $x = 4.7$ and $y = -2$

4 Using the formula $m = 2^x - y^3$, find the value of m if

a $x = 5$ and $y = 3$

b $x = 3$ and $y = -2$

c $x = -5$ and $y = \frac{1}{2}$

10 In the expansion of $(x+2)^n$, the coefficient of the x^3 term is two times the coefficient of the x^4 term. Find the value of n.

Review exercise

EXAM-STYLE QUESTIONS

1 Consider the arithmetic sequence 3, 7, 11, 15, ...

a Write down the common difference.

b Find u_{71} **c** Find the value of n such that $u_n = 99$

2 The first three terms of an infinite geometric sequence are 64, 16 and 4.

a Write down the value of r. **b** Find u_4.

c Find the sum to infinity of this sequence.

3 In an arithmetic sequence, $u_6 = 25$ and $u_{12} = 49$

a Find the common difference.

b Find the first term of the sequence.

4 Consider the arithmetic sequence 22, x, 38, ...

a Find the value of x. **b** Find u_{31}.

5 Evaluate the expression $\sum_{a=1}^{4}(3^a)$

6 Consider the geometric series 800 + 200 + 50 + ⋯

a Find the common ratio. **b** Find the sum to infinity.

7 Find all possible values of x such that this sequence is geometric: x, 12, $9x$, ...

EXAM-STYLE QUESTION

8 Find the x^3 term in the expansion of $(2x+3)^5$

9 A grocery store has a display of soup cans stacked in a pyramid. The top row has three cans, and each row has two more cans than the row above it.

a If there are 35 cans in the bottom row, how many rows are in the display?

b How many cans are in the display in total?

Review exercise

1 In an arithmetic series, the first term is 4 and the sum of the first 25 terms is 1000.

a Find the common difference. **b** Find the value of the 17th term.

EXAM-STYLE QUESTION

2 Consider the arithmetic sequence 3, 4.5, 6, 7.5, ...

a Find u_{63}. **b** Find the value of n such that $S_n = 840$

3 In an arithmetic series, the tenth term is 25 and the sum of the first 10 terms is 160.

a Find the first term and the common difference.

b Find the sum of the first 24 terms.

EXAM-STYLE QUESTIONS

4 In a geometric sequence, the first term is 3 and the sixth term is 96.

a Find the common ratio.

b Find the least value of n such that $u_n > 3000$

5 In an arithmetic sequence, the first term is 28 and the common difference is 50. In a geometric sequence, the first term is 1 and the common ratio is 1.5

Find the least value of n such that the nth term of the geometric sequence is greater than the nth term of the arithmetic sequence.

6 In a geometric series, the 3rd term is 45 and the sum of the first 7 terms is 2735.

Find the first term and the common ratio, r, if $r \in \mathbb{Z}$

EXAM-STYLE QUESTION

7 Find the term in x^4 in the expansion of $\left(\frac{x}{2}-3\right)^7$

8 In the expansion of $(ax + 2)^8$, the x^5 term has a coefficient of $\frac{7}{16}$.

Find the value of a.

9 At the beginning of 2010, the population of a country was 3.4 million.

a If the population grows at a rate of 1.6% annually, estimate the country's population at the beginning of 2040.

b If population growth continues at this rate, in what year would the population of the country be expected to exceed 7 million?

CHAPTER 6 SUMMARY

Patterns and sequences

- A **number sequence** is a pattern of numbers arranged in a particular order according to a rule.
- Each individual number, or element, of a sequence is called a **term**.

Arithmetic sequences

- In an arithmetic sequence, the terms increase or decrease by a constant value. This value is called the **common difference**, or d. The common difference can be a positive or a negative value.
- You can find the nth term of an arithmetic sequence using the formula:

  $u_n = u_1 + (n-1)d$

Continued on next

Geometric sequences

- In a **geometric sequence**, each term can be obtained by multiplying the previous term by a constant value. This value is called the **common ratio**, or r.
- You can find the nth term of a geometric sequence using the formula: $u_n = u_1(r^{n-1})$

Sigma (Σ) notation and series

- $\sum_{i=1}^{n} u_i$ means the sum of the first n terms of a sequence.

  You read this 'the sum of all the terms u_i from $i = 1$ to $i = n$'.

Arithmetic series

- You can find the sum of the first n terms of an arithmetic series using the formula:

  $S_n = \frac{n}{2}(u_1 + u_n)$ or $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

Geometric series

- You can find the sum of the first n terms of a geometric series using the formula:

  $S_n = \frac{u_1(r^n - 1)}{r-1}$ or $S_n = \frac{u_1(1-r^n)}{1-r}$, where $r \neq 1$.

Convergent series and sums to infinity

- For a geometric series with $|r|<1$, $S_\infty = \frac{u_1}{1-r}$

Pascal's triangle and the binomial expansion

- The number of combinations of n items taken r at a time is found by:

  $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, where $n! = n \times (n-1) \times (n-2) \times \ldots \times 1$

- The binomial theorem states that for any power of a binomial, where $n \in \mathbb{N}$,

  $$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}a^0 b^n$$

- You can also write the binomial expansion using sigma notation:

  $$(a+b)^n = \sum_{r=0}^{n}\left(\binom{n}{r}(a)^{n-r}(b)^r\right)$$